Yandex Research

Nesterov Finds GRAAL: Optimal and Adaptative Gradient Method for Convex Optimization

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Yandex Research

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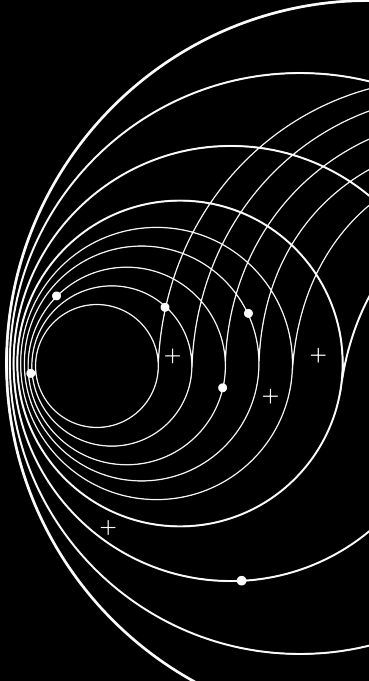

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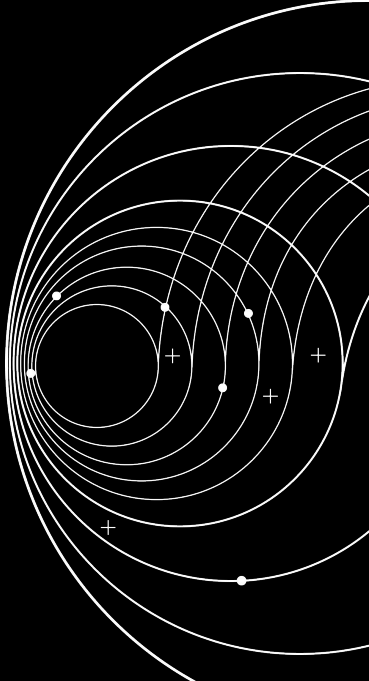
Introduction



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First-Order Optimization Methods

- Perform iterative updates using $f(x_k)$ and $\nabla f(x_k)$
- Simplest example – GD: $x_{k+1} = x_k - \eta \nabla f(x_k), \eta > 0$ – stepsize

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- It is not possible to obtain meaningful convergence guarantees with respect to the objective function gap, $f(x) - f^* \leq \epsilon$, in the general non-convex case; only first-order stationarity, $\|\nabla f(x)\| \leq \epsilon$, can be guaranteed

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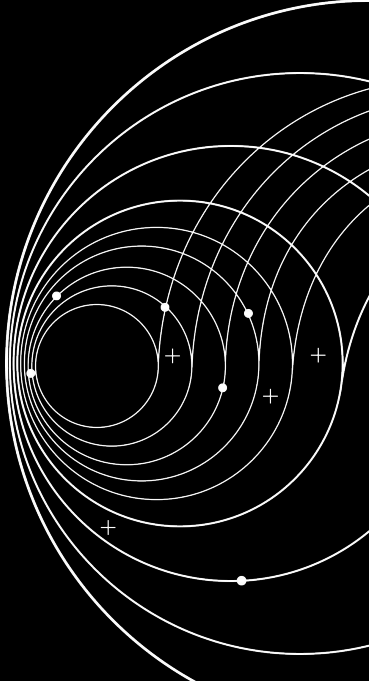
Adaptive Methods



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- Globally with hyperparameter tuning
- At each iteration with line-search

AdaGrad-type Methods

AdaGrad (scalar-stepsize vaiant)

(Duchi et al., 2011)

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Algorithms with Local Curvature Estimators

Local Curvature Estimator

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- local estimator of the inverse gradient Lipschitz constant

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GRAAL (Malitsky, 2020)

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Accelerated Algorithms with Local Curvature Estimators

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AC-FGM (Li and Lan, 2025)

$$
\begin{aligned}
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Issues with AC-FGM:

- **Allows only sub-geometric growth of stepsize:**

$$\eta_k \le k\eta_0$$

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$$\hat{x}_{k+1} = x_k - \eta_k\nabla f(\bar{x}_k)$$
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Local Curvature Estimator

$$\Lambda(x,x') = \frac{2\mathrm{D}_f(x;x')}{\|\nabla f(x) - \nabla f(x')\|}$$

- local estimator of the inverse gradient Lipschitz constant for convex functions

Issues with AC-FGM:

- **Allows only sub-geometric growth of stepsize:**

$$\eta_k \le k\eta_0$$

- GRAAL allows geometric growth of stepsize:

$$\eta_k \le (1+\gamma)^k\eta_0$$

GRAAL Stepsize

$$\eta_k = \min\left\{(1+\gamma)\eta_{k-1}, \nu\Lambda_k^2/\eta_{k-2}\right\}$$

Accelerated Algorithms with Local Curvature Estimators

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- **AdaNAG (Suh and Ma, 2025) has similar issues**

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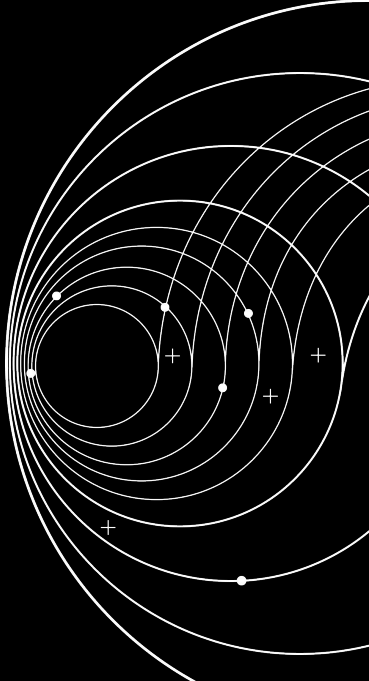
Yandex Research

GRAAL with Nesterov Acceleration



GRAAL Extrapolation and Nesterov Acceleration

Local Curvature Estimator

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GRAAL Extrapolation and Nesterov Acceleration

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Acceleration Framework (Borodich et al., 2025)

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- Replace objective function $f(x)$ with

$$f_k(x) = \alpha_k^{-1} f(\alpha_k x + (1 - \alpha_k)\bar{x}_k)$$

where $\bar{x}_k \in \mathbb{R}^d$ and $\alpha_k \in (0, 1]$

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GRAAL + Acceleration

$$\bar{x}_{k+1} = \alpha_k \hat{x}_k + (1 - \alpha_k)\bar{x}_k, \qquad \hat{x}_{k+1} = x_{k+1} + \theta(x_{k+1} - x_k), \qquad x_{k+1} = x_k - \eta_k \nabla f(\bar{x}_{k+1})$$

Problem: How to Choose η_k and α_k?

Local Curvature Estimator

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Issues with parameters η_k and α_k:

- **Not possible to choose α_k adaptively**
  cyclic dependency: $\eta_k \leftarrow \bar{x}_{k+1} \leftarrow \alpha_k \leftarrow \eta_k$
- Li and Lan (2025) choose $\alpha_k = 2/(k+2)$ in AC-FGM
  non-adaptive, implies sub-geometric growth of η_k

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Solution: Additional Coupling Step

Additional Coupling Step

- Update $\bar{x}_{k+1}$ as follows:

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where $\alpha_k, \beta_k \in (0, 1]$

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- replace $\nabla f(\bar{x}_{k+1}) \to \nabla f(\tilde{x}_k)$

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Accelerated GRAAL

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Solution: Additional Coupling Step

Choosing α_k, β_k, and η_k (from analysis)

Local Curvature Estimator

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$$\frac{\eta_k}{\alpha_k \beta_k} \leq \frac{\eta_{k-1}}{\alpha_{k-1}\beta_{k-1}} + \eta_k \leq \cdots \leq H_k = \sum_{i=0}^{k} \eta_i$$

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- Hence, choose $\beta_k = \eta_k / (\alpha_k H_k)$

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- Parameters α_k, β_k, and η_k satisfy the following:

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- Hence, choose $\beta_k = \eta_k / (\alpha_k H_k)$
- How to choose $\alpha_k \in (0, 1]$ and ensure $\beta_k \in (0, 1]$?

Local Curvature Estimator

$$\Lambda(x, x') = \frac{2\mathrm{D}_f(x; x')}{\|\nabla f(x) - \nabla f(x')\|}$$

Accelerated GRAAL

$$\begin{aligned}
\tilde{x}_k &= \alpha_k \hat{x}_k + (1 - \alpha_k)\bar{x}_k \\
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\hat{x}_{k+1} &= x_{k+1} + \theta(x_{k+1} - x_k) \\
x_{k+1} &= x_k - \eta_k \nabla f(\tilde{x}_k)
\end{aligned}$$

Solution: Additional Coupling Step

Choosing α_k, β_k, and η_k (from analysis)

- Parameters α_k, β_k, and η_k satisfy the following:

$$\frac{\eta_k}{\alpha_k \beta_k} \leq \frac{\eta_{k-1}}{\alpha_{k-1}\beta_{k-1}} + \eta_k \leq \cdots \leq H_k = \sum_{i=0}^{k} \eta_i$$

- Hence, choose $\beta_k = \eta_k / (\alpha_k H_k)$
- How to choose $\alpha_k \in (0, 1]$ and ensure $\beta_k \in (0, 1]$?
- Restrictions on stepsize η_k:

$$\eta_k \leq (1+\gamma)\eta_{k-1}$$

$$\eta_k \leq \frac{\nu H_{k-2}}{\eta_{k-2}} \cdot \min\{\Lambda(\bar{x}_k, \tilde{x}_k), \Lambda(\bar{x}_k, \tilde{x}_{k-1})\}$$

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- Choose $\alpha_k \in (0,1]$ adaptively as follows:

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- **Implementable: no cyclic dependencies**

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Accelerated GRAAL: Convergence Analysis

Accelerated GRAAL Stepsize

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Theorem 1 (Borodich and Kovalev, 2025)

There exist universal constants $\theta, \gamma, \nu > 0$ such that

$$\begin{aligned}\tfrac{1}{2}\|x_K - x\|^2 + H_{K-1}(f(\bar{x}_K) - f(x)) \\ \leq \tfrac{1}{2}\|x_0 - x\|^2 + \tfrac{(1+\gamma\theta)}{2}\eta_0^2\|\nabla f(x_0)\|^2\end{aligned}$$

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$$\eta_k \leq (1+\gamma)^k \eta_0$$

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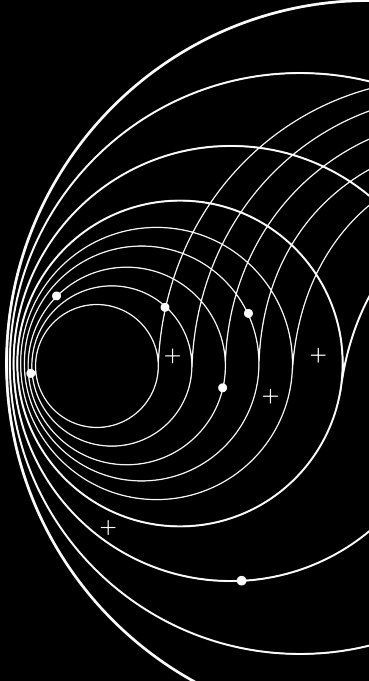
Yandex Research

Convergence Analysis for (L_0, L_1)-Smooth Functions



(L_0, L_1)-Smooth Functions

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(L_0, L_1)-Smooth Functions

(L_0, L_1)-Smoothness (Zhang et al., 2019)

$$\|\nabla^2 f(x)\| \leq L_0 + L_1 \|\nabla f(x)\|$$

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- Covers L-smooth functions ($L_0 = L, L_1 = 0$)

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- $x \mapsto \|x\|^p, \ p \geq 2$

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- $x \mapsto \|x\|^p, \ p \geq 2$
- $x \mapsto \exp(\langle a, x \rangle), \ a \in \mathbb{R}^d$

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Lemma 7 (Borodich and Kovalev, 2025)

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Lemma 5 (Borodich and Kovalev, 2025)

$$\sum_{i=1}^{K}(\eta_i \mathrm{D}_f(\bar{x}_i, \tilde{x}_i) + \eta_i^2\|\nabla f(x_i)\|^2) \leq \mathcal{D}^2$$

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- $\Lambda_k \geq \text{const}/\max\{L_1^2 \mathrm{D}_f(\bar{x}_k, \tilde{x}_k), L_1^2 \mathrm{D}_f(\bar{x}_{k-1}, \tilde{x}_{k-1})\}$

Lemma 5 (Borodich and Kovalev, 2025)

$$\sum_{i=1}^{K}(\eta_i \mathrm{D}_f(\bar{x}_i, \tilde{x}_i) + \eta_i^2\|\nabla f(x_i)\|^2) \leq \mathcal{D}^2$$

- Lemmas 5 and 7 allow to upper-bound the number of iterations where $\Lambda_k < \text{const}/L_0$

Local Curvature Estimator

$$\Lambda(x, x') = \frac{2\mathrm{D}_f(x; x')}{\|\nabla f(x) - \nabla f(x')\|}$$

Accel. GRAAL Parameters

$$\alpha_k = \frac{(1+\gamma)\eta_{k-1}}{H_{k-1}+(1+\gamma)\eta_{k-1}}$$

$$\Lambda_k = \min\{\Lambda(\bar{x}_k, \tilde{x}_k), \Lambda(\bar{x}_k, \tilde{x}_{k-1})\}$$

$$\eta_k = \min\left\{(1+\gamma)\eta_{k-1}, \frac{\nu H_{k-2}\Lambda_k}{\eta_{k-2}}\right\}$$

$$H_k = \sum_{i=0}^{k}\eta_i, \quad \beta_k = \frac{\eta_k}{\alpha_k H_k}$$

Key Lemmas

Lemma 7 (Borodich and Kovalev, 2025)

At least one of the following options holds:

- $\Lambda_k \geq \text{const}/L_0$
- $\Lambda_k \geq \text{const}/\max\{L_1\|\nabla f(\tilde{x}_k)\|, L_1\|\nabla f(\tilde{x}_{k-1})\|\}$
- $\Lambda_k \geq \text{const}/\max\{L_1^2 \mathrm{D}_f(\bar{x}_k, \tilde{x}_k), L_1^2 \mathrm{D}_f(\bar{x}_{k-1}, \tilde{x}_{k-1})\}$

Lemma 5 (Borodich and Kovalev, 2025)

$$\sum_{i=1}^{K}(\eta_i \mathrm{D}_f(\bar{x}_i, \tilde{x}_i) + \eta_i^2\|\nabla f(x_i)\|^2) \leq \mathcal{D}^2$$

- Lemmas 5 and 7 allow to upper-bound the number of iterations where $\Lambda_k < \text{const}/L_0$
- This allows to lower-bound H_k

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Main Result

Theorem 3 (Borodich and Kovalev, 2025)

Let $\eta_0 L_0 \exp(L_1\|x_0 - x^*\|) \leq 1$. Then $\mathcal{D} = \mathcal{O}(\|x_0 - x^*\|)$ and the following inequality holds:

$$H_K \geq \frac{c^2}{L_0}\Big[K - \big(1 + L_1^3\mathcal{D}^3\big) - \big(1 + L_1^2\mathcal{D}^2\big)\ln\Big[\tfrac{1}{\eta_0 L_0}\Big]\Big]^2$$

Local Curvature Estimator

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Corollary 3 (Borodich and Kovalev, 2025)

To reach precision $f(\bar{x}_K) - f^* \leq \epsilon$, the following number of iterations is sufficient:

$$K = \mathcal{O}\Big[1 + \sqrt{L_0\mathcal{D}^2/\epsilon} + L_1^3\mathcal{D}^3 + \big(1 + L_1^2\mathcal{D}^2\big)\ln\Big[\tfrac{1}{\eta_0 L_0}\Big]\Big]$$

Local Curvature Estimator

$$\Lambda(x, x') = \frac{2\mathrm{D}_f(x; x')}{\|\nabla f(x) - \nabla f(x')\|}$$

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$$\alpha_k = \tfrac{(1+\gamma)\eta_{k-1}}{H_{k-1}+(1+\gamma)\eta_{k-1}}$$
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$$H_k = \textstyle\sum_{i=0}^{k}\eta_i, \quad \beta_k = \tfrac{\eta_k}{\alpha_k H_k}$$

Comparison with AC-FGM and Ada-NAG

Corollary 3 (Borodich and Kovalev, 2025)

Let $\eta_0 L_0 \exp(L_1 \mathcal{D}) \leq 1$. Then $\mathcal{D} = \mathcal{O}(\|x_0 - x^*\|)$, and Accelerated GRAAL has the following iteration complexity:

$$K = \mathcal{O}\left[1 + \sqrt{L_0 \mathcal{D}^2/\epsilon} + L_1^3 \mathcal{D}^3 + \left(1 + L_1^2 \mathcal{D}^2\right) \ln\left[\frac{1}{\eta_0 L_0}\right]\right]$$

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Comparison with AC-FGM and AdaNAG:

- We can ensure $\eta_0 L_0 \exp(L_1 \|x_0 - x^*\|) \leq 1$ by choosing a very small η_0 at the cost of logarithmic factors

Comparison with AC-FGM and Ada-NAG

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- We can ensure $\eta_0 L_0 \exp(L_1 \|x_0 - x^*\|) \leq 1$ by choosing a very small η_0 at the cost of logarithmic factors
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- No results under (L_0, L_1)-smoothness for AC-FGM (Li and Lan, 2025) and AdaNAG (Suh and Ma, 2025)
- Adaptation at a geometric rate is crucial, as Λ_k may be exponentially small, $\Lambda_k \geq \exp(-3 L_1 \mathcal{D})$ (Borodich and Kovalev, 2025, Lemma 6)

Comparison with Algorithms for (L_0, L_1)-smooth Functions

Algorithm	Complexity	Remarks

Comparison with Algorithms for (L_0, L_1)-smooth Functions

Algorithm	Complexity	Remarks
Similar Triangles Method (Gasnikov and Nesterov, 2016) (Gorbunov et al., 2024)	$\sqrt{\frac{L_0\mathcal{D}^2}{\epsilon}} \times \sqrt{1 + L_1\mathcal{D}\exp(L_1\mathcal{D})}$	✘ Non-adaptive ✘ Not optimal

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AdGD (Malitsky and Mishchenko, 2020) (Gorbunov et al., 2024)	$\frac{L_0\mathcal{D}^2}{\epsilon} + (L_1\mathcal{D})^6$	✔ Adaptive ✘ Non-accelerated

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AdGD (Malitsky and Mishchenko, 2020) (Gorbunov et al., 2024)	$\frac{L_0\mathcal{D}^2}{\epsilon} + (L_1\mathcal{D})^6$	✔ Adaptive ✘ Non-accelerated
AGMsDR (Vankov et al., 2024)	$\sqrt{\frac{L_0\mathcal{D}^2}{\epsilon}} + (L_1\mathcal{D})^{5/3}$	✘ Non-adaptive ✔ Optimal

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AGMsDR (Vankov et al., 2024)	$\sqrt{\frac{L_0\mathcal{D}^2}{\epsilon}} + (L_1\mathcal{D})^{5/3}$	✘ Non-adaptive ✔ Optimal
AGD (Tyurin, 2025)	$\sqrt{\frac{L_0\mathcal{D}^2}{\epsilon}} + (L_1\mathcal{D})^2$	✘ Non-adaptive ✔ Optimal

Comparison with Algorithms for (L_0, L_1)-smooth Functions

Algorithm	Complexity	Remarks
Similar Triangles Method (Gasnikov and Nesterov, 2016) (Gorbunov et al., 2024)	$\sqrt{\frac{L_0\mathcal{D}^2}{\epsilon}} \times \sqrt{1 + L_1\mathcal{D}\exp(L_1\mathcal{D})}$	✘ Non-adaptive ✘ Not optimal
AdGD (Malitsky and Mishchenko, 2020) (Gorbunov et al., 2024)	$\frac{L_0\mathcal{D}^2}{\epsilon} + (L_1\mathcal{D})^6$	✔ Adaptive ✘ Non-accelerated
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AGD (Tyurin, 2025)	$\sqrt{\frac{L_0\mathcal{D}^2}{\epsilon}} + (L_1\mathcal{D})^2$	✘ Non-adaptive ✔ Optimal
Accelerated GRAAL (Borodich and Kovalev, 2025)	$\sqrt{\frac{L_0\mathcal{D}^2}{\epsilon}} + (L_1\mathcal{D})^3$	✔ Adaptive ✔ Optimal

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